Illustrating the Classification of Real Cubic Surfaces

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Summary. Knörrer and Miller classified the real projective cubic surfaces in $\mathbb{P}^3(\mathbb{R})$ with respect to their topological type. For each of their 45 types containing only rational double points we give an affine equation, s.t. none of the singularities and none of the lines are at infinity. These equations were found using classical methods together with our new visualization tool SURFEX. This tool also enables us to give one image for each of the topological types showing all the singularities and lines.

1 Introduction

A *projective real cubic surface in real projective three-space* $\mathbb{P}^3(\mathbb{R})$ is a homogenous polynomial f of degree 3 in four variables x, y, z, w with real coefficients:

$$f = \sum_{i,j,k,l \in \mathbb{N}_0 \ | \ i+j+k+l=3} a_{i,j,k,l} x^i y^j z^k w^l,$$

where $a_{i,j,k,l} \in \mathbb{R}$. In 1987, Knörrer and Miller [13] classified all such surfaces with respect to their topological type. A similar classification had already been given by Schläfli in the 19^{th} century [19], but Knörrer and Miller obtain more precise and more complete results. Some of these are based on ideas of Bruce and Wall [2] who gave a modern treatment of the complex case.

Here, we restrict ourselves to cubic surfaces with only rational double points which is the most interesting part of the classification. We summarize briefly Knörrer/Miller's main results on these surfaces and give an explicit real affine equation for each class in their list (see table 2 on page 7). These allow us to draw images for each class showing all singularities and lines (see fig. 2, 3, 4) using our software SURFEX [10].

In the already cited article, Schläfli also gave equations for each of his types and described their construction in a very geometric way. In many cases, it is easy to find real affine equations from these with the help of our tool SURFEX. But in the other cases, there are too many free parameters and we have to use other methods such as the deformation techniques described by Klein [11].

To perform these deformations explicitly, it is useful to have a visualization software at hand. We explain how to use our software SURFEX for such purposes. SURFEX can be used directly on our webpage [14]. It can also produce high quality raytraced images for publications in color or in black/white. All the images in the present paper are produced using SURFEX in connection with SINGULAR [9]. This computer algebra program was used to compute a primary decomposition of the ideal (f, F_9) describing the 27 lines of f with multiplicities which allowed us to draw the lines on the surfaces using SURFEX. Here, F_9 denotes Clebsch's covariant of degree 9 (see, e.g., [16, appendix 4.1] for a determinental formula for this covariant).

The webpage `www.CubicSurface.net` [15] contains some movies and more images. SURFEX [10] uses S. Endraß's SURF [7] to produce the high quality raytraced images of the surfaces and R. Morris's LSMP [17] and K. Polthier's JAVAVIEW [18] to allow rotation and scaling of a triangulated preview.

Several mathematicians have already given real affine equations for particularly interesting cubic surfaces such as the Clebsch Diagonal Surface or the four-nodal cubic surface. Recently, the architect J. Chertok collected equations for Rodenberg's 100-year-old series of plaster-models. These equations were communicated to him by different people, mainly S. Endraß and the second author. With these the architect recreated Rodenberg's series using $3d$-printers. But also Rodenberg's series is restricted to some types of cubic surfaces, and several of Rodenberg's models do not show all the projective real lines because some are at infinity. In fact, this was Rodenberg's intention. His aim was to give an overview of the possible singularities on cubic surfaces and the possible affine views of the projective surfaces. Here instead, we do not show different affine views of the same surfaces. We choose real affine equations that allow us to show all singularities and lines in a single image.

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2 The Main Results of Knörrer/Miller on Cubic Surfaces with only Rational Double Points

We briefly review some results of Knörrer and Miller. As already mentioned we restrict ourselves to those concerning only rational double points. In [13], the authors say that two cubic surfaces have the same topological type if they can be transformed continuously into each other without changing the shape. The precise definition uses the finer notion of equisingularity:

Definition 1 (4.1 in [13]).

1. *A differentiable family* $(Y_t)_{t\in[0,1]}$ *of cubic surfaces in* $\mathbb{P}^3(\mathbb{R})$ *with equations* f_t *is called* equisingular *if in the neighborhood of each point it can be extended to a family of diffeomorphisms of the surrounding space. I.e., if for each* $t_0 \in [0,1]$ *and each* $p \in Y_{t_0}$ *there exists a neighborhood* I *of* t_0 *in* $[0,1]$*, a neighborhood* U *of* p *in* $\mathbb{P}^3(\mathbb{R})$ *and a diffeomorphism* $\Phi : U \times I \to U \times I$*, s.t. the following diagram commutes:*

$$\begin{array}{ccc} U \times I & \overset{\Phi}{\to} & U \times I \\ \cup & & \cup \\ \{(x,t) \in U \times I \mid f_{t_0}(x) = 0\} & \overset{\Phi}{\to} & \{(x,t) \in U \times I \mid f_t(x) = 0\} \\ & pr_2 \searrow \quad \swarrow pr_2 & \\ & I & \end{array}$$

Two surfaces that can be transformed into each other by an equisingular family are called equisingular isotopic.
2. *Two cubic surfaces* $Y_0, Y_1 \in \mathbb{P}^3(\mathbb{R})$ *have* the same topological type *if there are projectively equivalent surfaces* Y_0', Y_1' *which are equisingular isotopic.*

There exist at most two different equisingular isotopy classes of cubic surfaces of the same topological type. Equisingular families are characterized by their configuration of singularities:

Proposition 1 (4.2 in [13]).

1. *If two cubic surfaces in* $\mathbb{P}^3(\mathbb{R})$ *with only isolated singularities have the same topological type then suitable neighborhoods of their singular sets are analytically isomorphic.*
2. *Any differentiable family of cubics in* $\mathbb{P}^3(\mathbb{R})$ *with only isolated singularities for which the configuration of singularities is constant is equisingular.*

Table 1 on the next page gives an overview of the rational double points occuring on cubic surfaces (see also [5] or [1]). The classical geometers associated to each surface f a *class* which is the number of tangency points f has with a generic pencil of hyperplanes (for computing the class see [2, sect. 3]). The subscript of the old names for the singularities is the number by which the class drops when a cubic surface possesses such a singularity (see table 1).

For the following definition we assume familiarity with some concepts from algebraic geometry, in particular with the blowup. A reader who is not familiar with this should simply read the numbers from table 1 on the following page.

Here, we just want to mention that it is well-known that the blowup $\widetilde{\mathbb{P}^2(\mathbb{C})}$ of the projective plane $\mathbb{P}^2(\mathbb{C})$ in a set of six points Σ (*basepoints*) which are in general position (i.e. no three on a line, no six on a conic) is a smooth complex cubic surface and that all smooth complex cubic surfaces can be obtained in this way. This blowup is a birational map which is a bijection away from the basepoints, i.e. for all points in $\mathbb{P}^2(\mathbb{C})\backslash\Sigma$. In the real case, we have to be more careful: the cubic F_5 with two components is not the result of such a blowup:

Definition 2 (p. 54/55 in [13]).

1. *$\mu_{\mathbb{R}}$ denotes the number of (-2)-curves defined over $\mathbb{R}$ in the dual resolution graph of a rational double point that is defined over $\mathbb{R}$. ν denotes the number of pairs of non-intersecting complex conjugate (-2)-curves in this graph.*

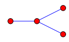
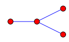
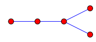
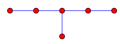
Name	Old Name	Normal Form	Coxeter Diagram	$\mu_{\mathbb{R}}$	ν	
A_{2k}^-	B_{2k+1}	$x^{2k+1}+y^2-z^2$		$2k$	0	$k=1,2$
A_{2k}^+	B_{2k+1}	$x^{2k+1}+y^2+z^2$		0	$k-1$	$k=1$
A_{2k-1}^-	B_{2k}	$x^{2k}+y^2-z^2$		$2k-1$	0	$k=2,3$
A_{2k-1}^+	B_{2k}	$x^{2k}-y^2-z^2$		1	$k-1$	$k=2$
A_1^-	C_2	$x^2+y^2-z^2$		1	0	
$A_1^{\bullet}$	C_2	$x^2+y^2+z^2$		1	0	
D_4^-	U_6	$x^2y-y^3-z^2$		4	0	
D_4^+	U_6	$x^2y+y^3+z^2$		2	1	
D_5^-	U_7	$x^2y+y^4-z^2$		5	0	
E_6^-	U_8	$x^3+y^4-z^2$		6	0	

Table 1. The types of singularities occuring on real cubic surfaces, their normal forms, and the numbers $\mu_{\mathbb{R}}$ and ν. For later use, we also give their Coxeter Diagrams.

2. *Let Σ be a sequence of six points defined over $\mathbb{R}$ in* almost general position in *$\mathbb{P}^2(\mathbb{C})$ in the sense of [4, p. 39]. Then there exists $r(\Sigma) \in \mathbb{N}_0$, s.t. Σ consists of $2r$ points that are invariant under complex conjugation and $6-2r$ pairwise compl. conj. points. We call $r(\Sigma)$ the* reality index of *Σ.*
3. *Let X be a cubic surface in $\mathbb{P}(\mathbb{C})$ defined over $\mathbb{R}$ with only rational double points. The* reality index $r(X)$ of *X is defined as follows: Let $\widetilde{X}$ denote the desingularization of X and $\overline{X}(\Sigma)$ the blowup of $\mathbb{P}^2(\mathbb{C})$ along Σ. Then, $r(X) = r(\Sigma)$, if $\widetilde{X} \cong \overline{X}(\Sigma)$ for a sequence Σ of six points in almost general position in $\mathbb{P}^2(\mathbb{C})$. Otherwise, $r(X) = -1$.*

Theorem 1 (Satz 2.8 in [13]). *Let $X \subset \mathbb{P}^3(\mathbb{C})$ be a cubic surface defined over $\mathbb{R}$ with only rational double points as singularities. Suppose that the real part $X_{\mathbb{R}} \subset \mathbb{P}^3(\mathbb{R})$ of X has k singular points. Denote by $\mu_{\mathbb{R}}(X)$ the sum of the $\mu_{\mathbb{R}}$ for these singular points and by $\nu(X)$ the sum of the ν of all singularities on X. Then the real part $X_{\mathbb{R}}$ contains exactly $l(X_{\mathbb{R}})$ lines, where*

$$l(X_{\mathbb{R}}) = \frac{(2+2r(X)-\mu_{\mathbb{R}}(X))(1+2r(X)-\mu_{\mathbb{R}}(X))}{2} - (r(X)-2) + k - \nu(X).$$

For a cubic surface $X \subset \mathbb{P}^3(\mathbb{C})$ we can read the topology of its real part $X_{\mathbb{R}} \subset \mathbb{P}^3(\mathbb{R})$ from the reality index. E.g., the five smooth cubic surfaces, classically denoted by $F_1, F_2, \ldots, F_5$ (see [20]), are classified by the reality index, e.g., $r(F_5) = -1$. Here is another result of Knörrer/Miller of this kind:

Lemma 1 (3.2, 3.3, 4.3 in [13]).

1. *If $X_{\mathbb{R}}$ does not contain any singularity of type $A^{\bullet}$ and $r(X) \geq 0$ then $X_{\mathbb{R}}$ is connected and $\chi(X_{\mathbb{R}}) = 1 - 2r(X) + \mu_{\mathbb{R}}(X)$. If $r(X) = -1$ then $X_{\mathbb{R}}$ is diffeomorphic to the disjoint union $\mathbb{P}^2(\mathbb{R}) \sqcup S^2$.*
2. *If $p \in X_{\mathbb{R}}$ is a singular point of type $A^{\bullet}$ then $X_{\mathbb{R}}$ does not contain any other singularity and $X_{\mathbb{R}}$ is diffeomorphic to $\mathbb{P}^2(\mathbb{R}) \sqcup \{p\}$.*
3. *Cubic surfaces of the same topological type are homeomorphic.*
4. *Two cubic surfaces of the same topological type with only rational singularities have the same reality index.*

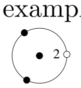
The following is Knörrer/Miller's main result on cubic surfaces with only rational double points:

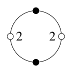
Theorem 2 (Classification, Liste 4 in [13]). *Let $X \subset \mathbb{P}^3(\mathbb{C})$ be a cubic surface defined over $\mathbb{R}$ with only rational double points and let $X_{\mathbb{R}} = X \cap \mathbb{P}^3(\mathbb{R})$ be its real part. Then the topological type of $X_{\mathbb{R}}$ is one of the 45 types given in table 2 on page 7. If X has exactly $3A_1^-$ singularities and X contains exactly 12 lines (no. 18/19 in the table) then its topological type can be determined by prop. 2 below. Otherwise, the topological type of X is determined by its singularities, its number of lines, and the reality index $r(X)$.*

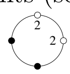
To explain how to distinguish between the types 18 and 19, we need Knörrer/Miller's notion of a *configuration type of an A_1^- singularity.* We only give a sloppy definition and illustrate it using SURFEX, see [13, p. 63] for details. For this local study we have to work in affine space:

Recall that the *tangent cone $tc(f)$* of a singularity f at the origin is the lowest non-zero homogenous part of f. For an A_1^- singularity, it is a cone of the form $x^2+y^2-z^2$. The tangent cone intersects the cubic surface X in a curve of degree $2 \cdot 3 = 6$, which consists in fact of six lines, counted with multiplicities. Knörrer/Miller describe such a configuration by a small circle together with six points (counted with multiplicities), because a small real sphere around the singularity intersects X in two small real „circles" (fig. 1 on the following page). On each of these circles there lies one point of each of the real lines. Therefore, Knörrer/Miller denote a pair of complex conjugated lines by a point in the center of the circle, the real points are drawn on the circle in the correct order. Different such configurations correspond to cubic surfaces of different topological types.

Example 1. Example (a) is a configuration with one real point of multiplicity 2, two real ones of multiplicity 1, and two complex conjugated ones. The other two examples show two doubled and two simple points (see fig. 1):

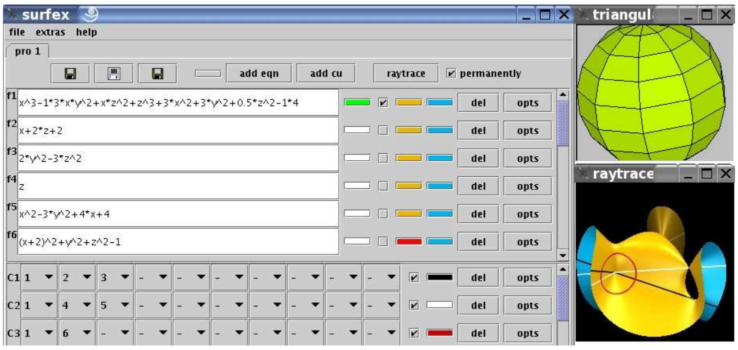
(a) 

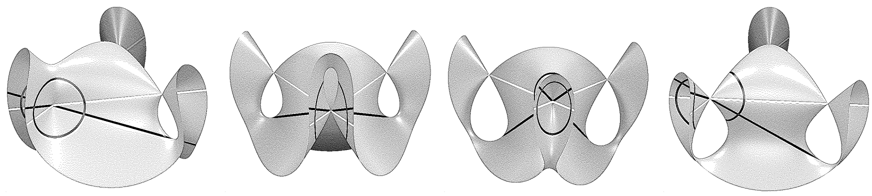
, (b) 

(KM$_{18}$ in fig. 1), (c) 

(KM$_{19}$ in fig. 1). □

Proposition 2 (Topological Types 18/19, p. 63 in [13]). *If a cubic surface X has exactly $3A_1^-$ singularities and contains 12 lines then X has the*





KM$_{18}$ (a) KM$_{18}$ (b) KM$_{19}$ (a) KM$_{19}$ (b)

Fig. 1. The configuration of the lines cut out by the tangent cone at one of the three A_1^- singularities of our surfaces with topological types no. 18 and 19. For each of the surfaces, we show two views (a), (b) from different angles. The white lines have multiplicity two, the black ones have multiplicity one. The figure above illustrates how SURFEX can draw curves on surfaces using the corresponding feature of SURF. To draw the two doubled white lines, we computed the equations `f4`, `f5` cutting these out on the surface using SINGULAR. Then we chose the numbers of the equations from the drop down menu in the row called `C2` and selected the color white.

topological type 18 *if the singular points have a configuration of type* *(example 1 (b)). Otherwise, the* A_1^- *singularities of* X *have a configuration of type* *(example 1 (c)) and* X *has the topological type* 19.

3 Constructing Nice Real Affine Equations

3.1 Nice Equations

By a *nice* real affine equation f for a given topological type t we mean an equation, s.t. its projective closure $\overline{f}$ has the required topological type and s.t. the plane at infinity neither contains a singularity nor a line of $\overline{f}$. It has also to be possible to see all its singularities and lines in a single picture (modulo guessing using symmetries). This is not a precise definition. Nevertheless, we formulate our main result in the form of a theorem:

Name	*Sp.*	*Cl.*	*Sing.*	r	l	Equation
KM_1	I	12	$\emptyset$	3	27	$\mathrm{KM}_{27} + \frac{3}{2}(x^2+y^2-z^3)$
KM_2	I	12	$\emptyset$	2	15	$\mathrm{KM}_{27} + \frac{8}{5}((z+1)^2-z^2)$
KM_3	I	12	$\emptyset$	1	7	$\mathrm{KM}_{27} + \frac{2}{3}((z+1)^2+(x-1)^2)-4y^2$
KM_4	I	12	$\emptyset$	0	3	$\mathrm{KM}_2 - 4$
KM_5	I	12	$\emptyset$	-1	3	$\mathrm{KM}_{27} - \frac{2}{3}((z+1)^2+z^2)$
KM_6	II	10	A_1^-	3	21	$\mathrm{KM}_{27} + 2(x^2+y^2)$
KM_7	II	10	A_1^-	2	11	$\mathrm{KM}_{27} + z^3 + y^2$
KM_8	II	10	A_1^-	1	5	$\mathrm{KM}_6 - 4y^2$
KM_9	II	10	A_1^-	0	3	$\mathrm{KM}_6 - 3(x^2+y^2)$
KM_{10}	II	10	$A_1^{\bullet}$	0	3	$pc + (z+1)\cdot z^2$
KM_{11}	IV	8	$2A_1^-$	3	16	$\mathrm{KM}_{27} + y^2$
KM_{12}	IV	8	$2A_1^-$	2	8	$\mathrm{KM}_{27} + z^2 - \frac{1}{5}(x+\frac{1}{2})^2$
KM_{13}	IV	8	$2A_1^-$	1	4	$\mathrm{KM}_{27} - y^2$
KM_{14}	III	9	A_2^-	3	15	$\mathrm{KM}_{21} + \frac{1}{10}(y-1)^2$
KM_{15}	III	9	A_2^-	2	7	$pl + z^3 - z^2(x-1) - \frac{1}{5}(x-y)^2$
KM_{16}	III	9	A_2^-	1	3	$\mathrm{KM}_{43} - y^2$
KM_{17}	III	9	A_2^+	0	3	$pc + z^3$
KM_{18}	VIII	6	$3A_1^-$	3	12	$\mathrm{KM}_{43} + z^2(x+\frac{1}{2})$
KM_{19}	VIII	6	$3A_1^-$	3	12	$\mathrm{KM}_{43} + 2z^2$
KM_{20}	VIII	6	$3A_1^-$	2	6	$\mathrm{KM}_{27} - z^2$
KM_{21}	VI	7	$A_2^-A_1^-$	3	11	$pl + z^3 + z^2(x+y-2) + \frac{1}{10}(x-1)^2$
KM_{22}	VI	7	$A_2^-A_1^-$	2	5	$pl + z^3 + z^2(x+y) + \frac{1}{5}(x-1)^2$
KM_{23}	V	8	A_3^-	3	10	$wxy + (x+z)(y^2 - (\frac{2}{3}x)^2 - (\frac{3}{5}z)^2)$, $w = 1-x$
KM_{24}	V	8	A_3^-	2	4	$\mathrm{KM}_{32} - \frac{1}{100}z^2(x-z)$
KM_{25}	V	8	A_3^-	1	2	$\mathrm{KM}_{32} + \frac{1}{100}z^2(x-z)$
KM_{26}	V	8	A_3^+	1	4	$2(x^2+y^2)w + 2x(z^2-2x^2-4y^2)$, $w = 1-y$
KM_{27}	XVI	4	$4A_1^-$	3	9	$4(pc+\frac{1}{2}) + 3(x^2+y^2)(z-6) - z(3+4z+7z^2)$
KM_{28}	XIII	5	$A_2^-2A_1^-$	3	8	$\mathrm{KM}_{43} + z^2(x+2)$
KM_{29}	IX	6	$2A_2^-$	3	7	$\mathrm{KM}_{43} + (x-1)z$
KM_{30}	IX	6	$2A_2^-$	2	3	$\mathrm{KM}_{43} - \frac{3}{10}(x-1)^2$
KM_{31}	X	6	$A_3^-A_1^-$	3	7	$wxz - (x+z)(x^2-y^2)$, $w = 1-z$
KM_{32}	X	6	$A_3^-A_1^-$	2	3	$wxy - (x+z)(x^2+y^2)$, $w = 1-z$
KM_{33}	VII	7	A_4^-	3	6	$wxy + y^2z + yx^2 - z^3$, $w = 1-x-y-z$
KM_{34}	VII	7	A_4^-	2	2	$wxy - y^2z + yx^2 - z^3$, $w = 1-x-y-z$
KM_{35}	XII	6	D_4^-	3	6	$(x+y+z)^2w + xyz$, $w = \frac{1}{2}(1-x-y-z)$
KM_{36}	XII	6	D_4^+	1	2	$(x+y+z)^2w + (x^2+y^2)z$, $w = \frac{1}{2}(1-x-y-z)$
KM_{37}	XVII	4	$2A_2^-A_1^-$	3	5	$\mathrm{KM}_{43} + (x-1)z^2$
KM_{38}	XVIII	4	$A_3^-2A_1^-$	3	5	$wxz + y^2(x+z)$, $w = 2(1+x-y+z)$
KM_{39}	XIV	5	$A_4^-A_1^-$	3	4	$wxz - y^2z + \frac{1}{2}x^2y$, $w = \frac{1}{8}(1-y-z)$
KM_{40}	XI	6	A_5^-	3	3	$wxz + y^2z + x^3 - z^3$, $w = 1-x$
KM_{41}	XI	6	A_5^-	2	1	$wxz + y^2z + x^3 + z^3$, $w = 1$
KM_{42}	XV	5	D_5^-	3	3	$wx^2 + y^2z + xz^2$, $w = 1+x$
KM_{43}	XXI	3	$3A_2^-$	3	3	$tl + z^3$
KM_{44}	XIX	4	$A_5^-A_1^-$	3	2	$wxz - y^2z - x^3$, $w = 1-z$
KM_{45}	XX	4	E_6^-	3	1	$x^2w - xz^2 + y^3$, $w = 1-x-y$

Table 2. Our nice real affine equations for Knörrer/Miller's 45 topological types. The abreviation *Sp.* denotes Schläfli's *species* of the surface, *Cl.* its class, *Sing.* its singularities. r denotes the reality index and l the number of real lines on the surface.

1–3:
–
$(r{:}3, 27l.)$
–
$(r{:}2, 15l.)$
–
$(r{:}1, 7l.)$

4–6:
–
$(r{:}0, 3l.)$
–
$(r{:}\text{-}1, 3l.)$
A_1^-
$(r{:}3, 21l.)$

7–9:
A_1^-
$(r{:}2, 11l.)$
A_1^-
$(r{:}1, 5l.)$
A_1^-
$(r{:}0, 3l.)$

10–12:
$A_1^{\bullet}$
$(r{:}0, 3l.)$
$2A_1^-$
$(r{:}3, 16l.)$
$2A_1^-$
$(r{:}2, 8l.)$

13–15:
$2A_1^-$
$(r{:}1, 4l.)$
A_2^-
$(r{:}3, 15l.)$
A_2^-
$(r{:}2, 7l.)$

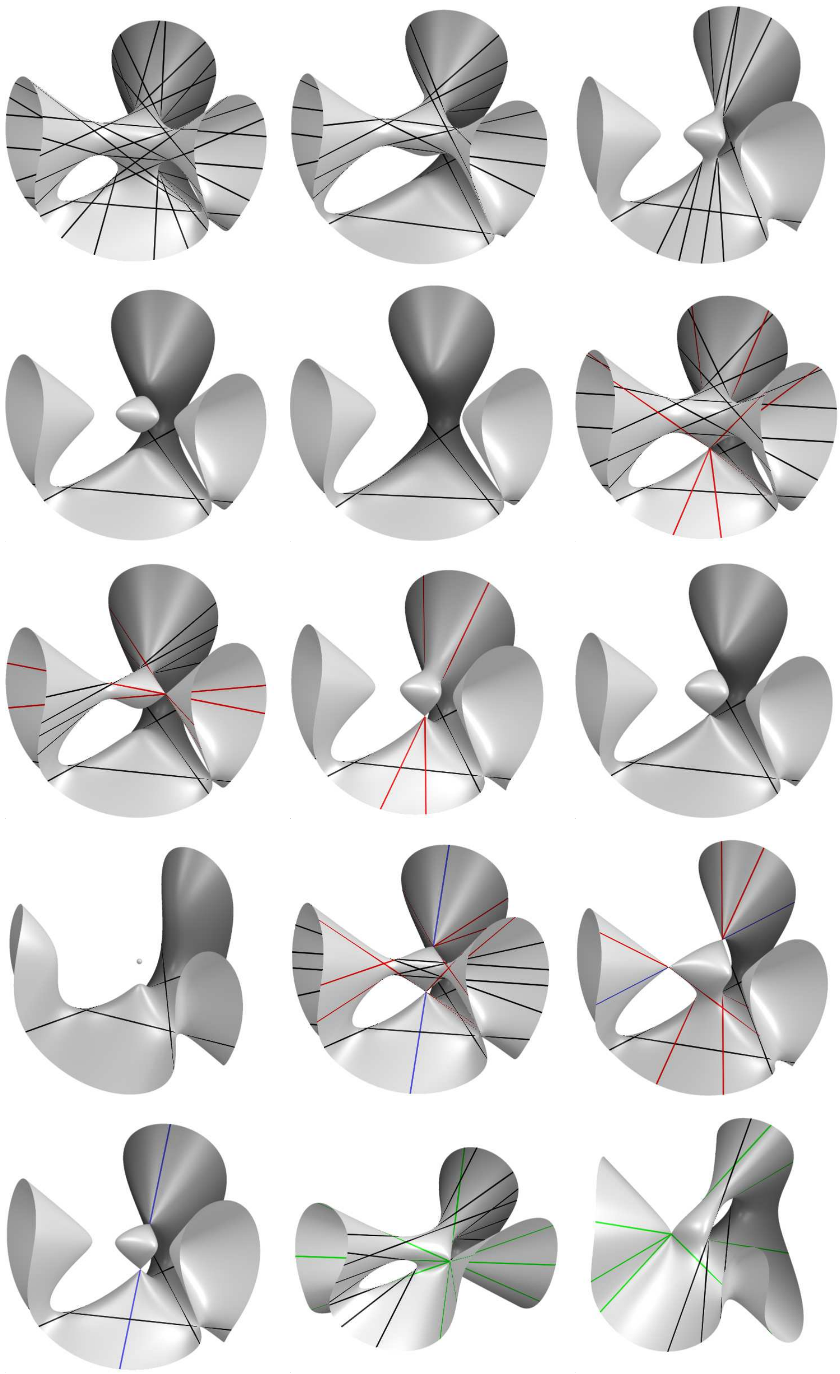


Fig. 2. The surfaces $\mathrm{KM}_1, \dots, \mathrm{KM}_{15}$. The colors of the lines indicate their multiplicities: ■ 1, ■ 2, ■ 3, ■ 4, ■ 5, ■ 6, ■ 8, ■ 9, ■ 10, ■ 12, ■ 15, ■ 16, ■ 27.

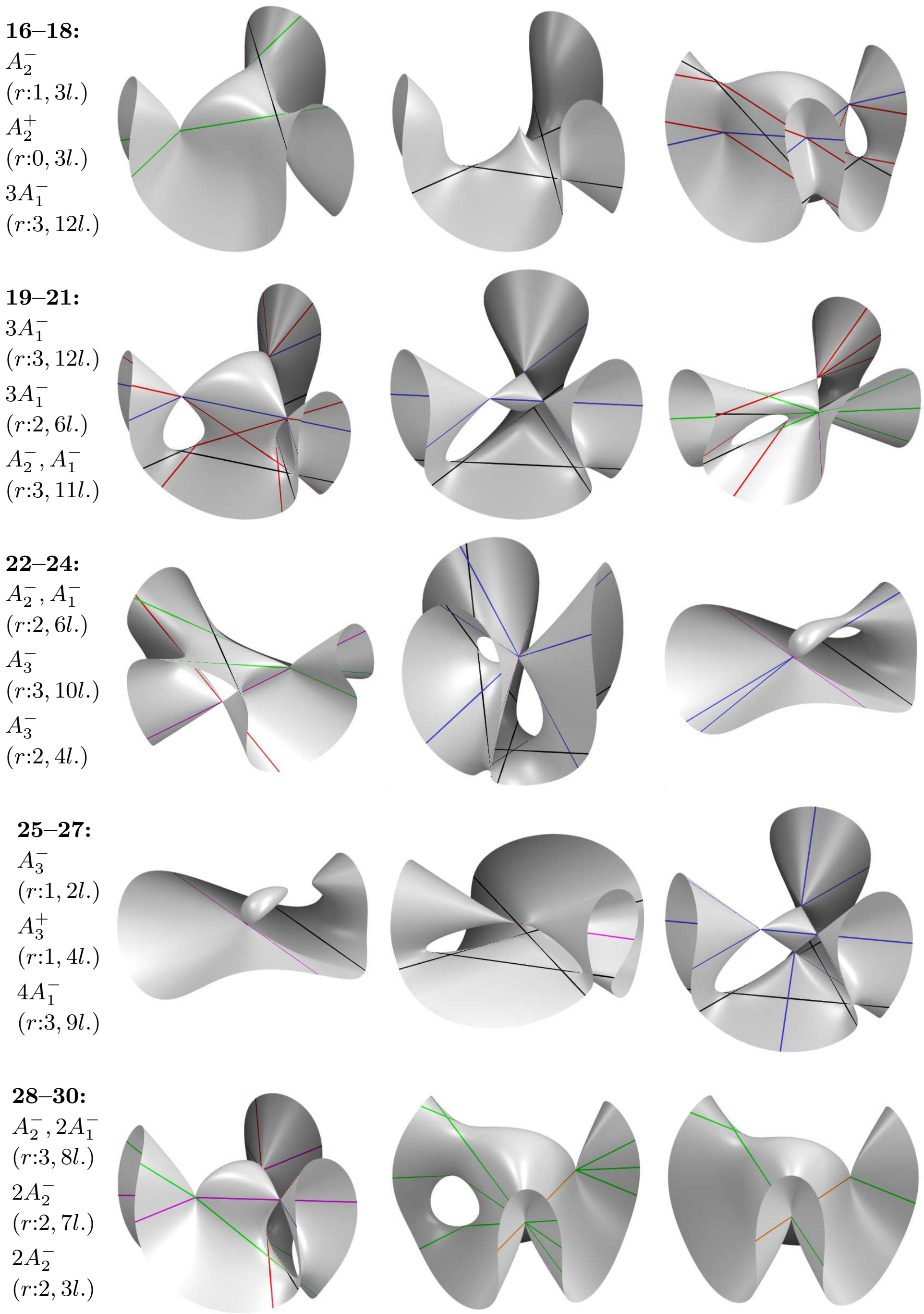
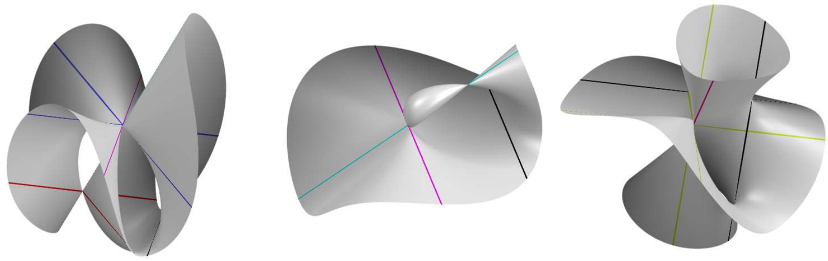
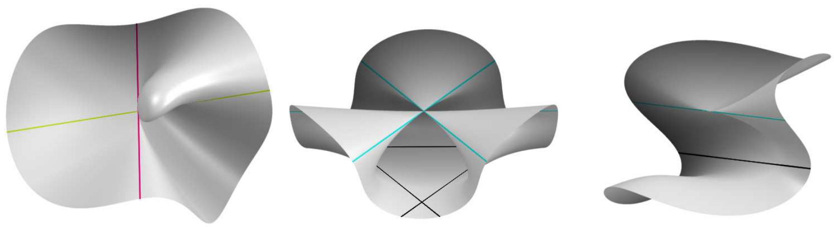


Fig. 3. The surfaces $\mathrm{KM}_{16}, \ldots, \mathrm{KM}_{30}$. The colors of the lines indicate their multiplicities: 1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 16, 27.

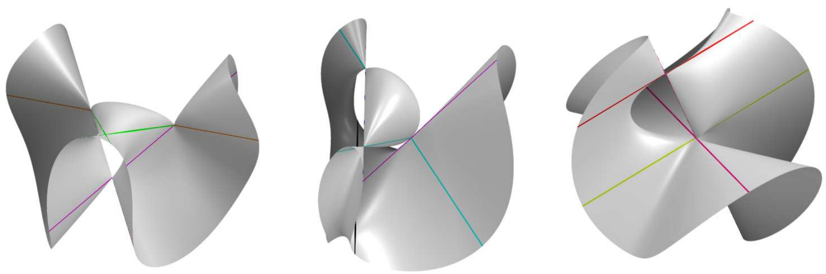
31–33:
A_3^-, A_1^-
$(r{:}3, 7l.)$
A_3^-, A_1^-
$(r{:}2, 3l.)$
A_4^-
$(r{:}3, 6l.)$



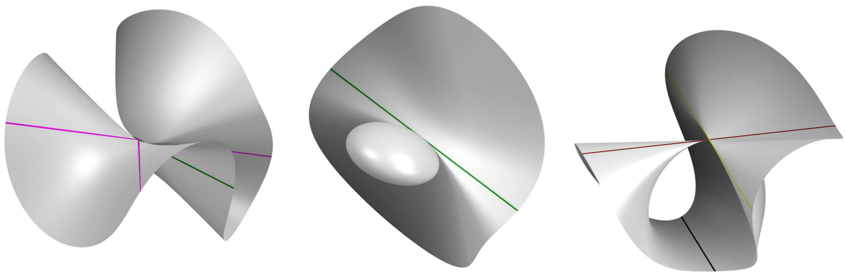
34–36:
A_4^-
$(r{:}2, 2l.)$
D_4^-
$(r{:}3, 6l.)$
D_4^+
$(r{:}1, 2l.)$



37–39:
$2A_2^-, A_1^-$
$(r{:}3, 5l.)$
$A_3^-, 2A_1^-$
$(r{:}3, 5l.)$
A_4^-, A_1^-
$(r{:}3, 4l.)$



40–42:
A_5^-
$(r{:}3, 3l.)$
A_5^-
$(r{:}2, 1l.)$
D_5^-
$(r{:}3, 3l.)$



43–45:
$3A_2^-$
$(r{:}3, 3l.)$
A_5^-, A_1^-
$(r{:}3, 2l.)$
E_6^-
$(r{:}2, 1l.)$

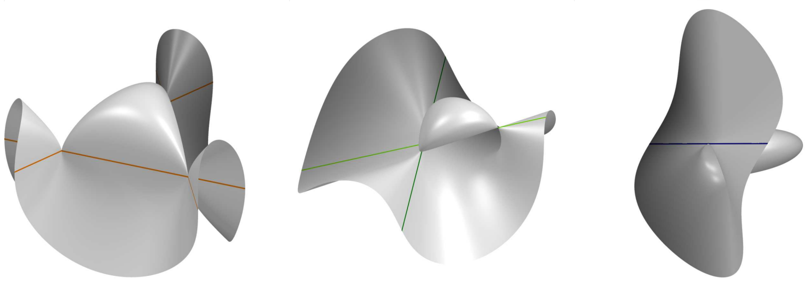


Fig. 4. The surfaces $\mathrm{KM}_{31}, \ldots, \mathrm{KM}_{45}$. The colors of the lines indicate their multiplicities: ■ 1, ■ 2, ■ 3, ■ 4, ■ 5, ■ 6, ■ 8, ■ 9, ■ 10, ■ 12, ■ 15, ■ 16, ■ 27.

Theorem 3. *For each topological type* $t \in \{1, 2, \ldots, 45\}$ *of cubic surfaces with only rational double points there is a nice affine equation* KM_t *in the sense of the preceding paragraph. The equations* KM_t *are given in table 2 on page 7 and the corresponding pictures are shown in the figures 2, 3, 4.*

Remark 1. For a nice equation for a given topological we do not require the greatest possible symmetry because we want the equations to be generic in the sense that the configuration of the lines on the surface should not be too special. E.g., the Clebsch Cubic Surface has 10 so-called *Eckardt Points* in which three of its 27 real lines meet, but a generic cubic surface with 27 lines does not have any such point.

Remark 2. Schläfli orders the cubic surfaces first by their class and then by the worst singularity occuring. This differs from Knörrer/Miller's order which is first by the sum of the Milnor numbers of the singularities and then by the worst singularity occuring.

In the following subsections we describe how to construct such surfaces.

3.2 Via Projective Equations

For the projective case, Schläfli already gave equations in [19]. He describes in a very geometric way how to construct them. In [3], Cayley gives the same equations again and computes a lot of additional data connected to the surfaces.[3]

To obtain a nice real affine equation from one of Schläfli's equations is an easy task for most topological types with higher singularities (A_3 or higher): We just have to choose a good hyperplane at infinity and maybe some constants which is not difficult using our tool SURFEX:

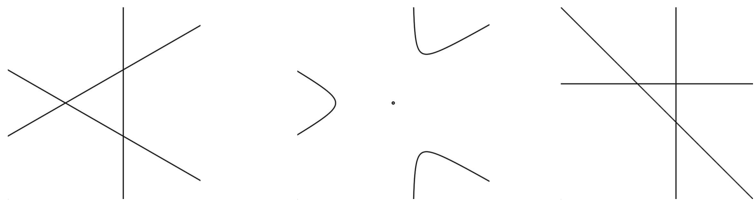
Example 2. Let us take the equation $wxz + y^2z + x^3 = 0$ given by Schläfli [19, p. 357] for a projective cubic surface with an A_1 and A_5 singularity. The choice $w = 1 - z$ gives our affine equation KM_{44}.

For those surfaces with only A_1 and A_2 singularities, this method does not work well because of the great number of free parameters. In this case, we can either write down the equation directly (section 3.3), or we can use a deformation process (section 3.4) already described by F. Klein in [11].

3.3 Direct Construction

In some cases, it is easy to write down a nice real affine equation for a topological type directly using symmetry. For this purpose, we will use the three plane curves shown in figure 5.

[3] Attention, Cayley's list on p. 321 contains some typos.



$tl := x^3 + 3x^2 - 3xy^2 + 3y^2 - 4$ $pc := tl + 4$ $pl := (x-1)(y-1)(x+y)$

Fig. 5. Three plane curves, useful for constructing nice equations for cubic surfaces.

Example 3 (Constructing KM_{43} *with three* A_2^- *Singularities).* We take the polynomial tl defining three triangle-symmetric lines (fig. 5) in the x, y-plane and add the term z^3: $\mathrm{KM}_{43} = tl + z^3$. At each intersection point of the lines tl, this gives a singularity of type A_2^- with z-coordinate 0, see fig. 8(a).

The four-nodal surface KM_{27} can be constructed in a similar way. This and a lot more information on nodal surfaces with dihedral symmetry can be found in S. Endraß's Ph.D. thesis [6]. The following example uses a plane curve with a solitary point. In the same way we obtain the surface KM_{26} with an A_3^+ singularity.

Example 4 (Constructing KM_{10} *with an* $A_1^{\bullet}$ *Singularity).* To construct a surface with an $A_1^{\bullet}$ Singularity which has the normal form $x^2 + y^2 + z^2$ we start with the triangle-symmetric plane cubic pc (fig. 5). The origin is a solitary point (i.e., a singularity with normal form x^2+y^2). Thus the surface $pc+z^2$ has an $A_1^{\bullet}$ singularity with normal form $x^2+y^2+z^2$ and is triangle-symmetric. To obtain the desired affine topology we require a third root on the $\{x = y = 0\}$ axes at $z = -1$: $\mathrm{KM}_{10} = pc + (z+1)\cdot z^2$.

3.4 The Deformation Process

Klein's strategy for obtaining surfaces with fewer singularities from surfaces with many singularities is based on the fact that any singularity on a cubic surface can be deformed separately.

In the case of complex projective cubic surfaces, this fact can be formulated in the following way (see Knörrer/Barth's article in [8] for an overview on this and other visible properties of cubic surfaces): The configurations of rational double points occuring on cubic surfaces are exactly those for which the disjoint union of their Coxeter Diagrams is a subgraph of the Coxeter Diagram of $\widetilde{E}_6$, see fig. 6. A surface can be specialized into another one if and only if its graph is contained in the other's graph.

By the definition of a singularity, the origin can only be a singularity of an affine surface f if the tangent cone of f has degree at least 2. Thus, in order to smooth an isolated singularity at the origin, we can simply add a term of degree 1 or 0.

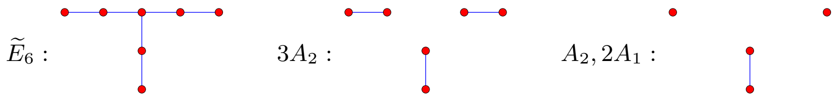


Fig. 6. There exists a cubic surface with three singularities of type A_2, because the disjoint union of three graphs of the A_2 singularity is a subgraph of the graph $\widetilde{E}_6$. A cubic surface with an A_2 singularity and two A_1 singularities can be specialized into one with three A_2 singularities as can be seen from the graphs. See table 1 on page 4 for the Coxeter Diagrams of the singularities of the rational double points on cubic surfaces.

But which terms can we add to the equation of f without changing the type of a singularity at the origin? For A_1 singularities, this is very easy: These singularities are characterized by the fact that their tangent cone also defines an A_1 singularity.[4] So, we can add any term of degree greater than two and any term of degree two whose coefficient is small enough. E.g. $x^2 + y^2 - z^2 + \frac{1}{10}z^2 + \frac{1}{13}xy + x^3$ has a singularity of type A_1^- at the origin.

Using the preceding facts we can deform a cubic surface with four singularities of type A_1^- into one with only three such singularities:

Example 5 (Smoothing one of four A_1 Singularities). Let KM_{27} be the cubic surface with four A_1^--singularities (see table 2 on page 7). Three of its singularities lie in the plane $\{z = 0\}$. Using SURFEX, it is easy to find an ε, s.t. the surface $\mathrm{KM}_{27} + \varepsilon z^2$ has the desired topology (see fig. 7):

Go to the SURFEX web-page [10], start the SURFEX program, and enter the equation of KM_{27}. Then add a term `+0.1*z^2` and check the `permanently` checkbox – this will premanently recompute raytraced images of your surface. Drag the computer mouse over the green ball to rotate the surface until you see all singularities. You can scale the image by pressing `s` on your keyboard while dragging. Now your SURFEX screen should look similar to fig. 7 on the following page. The singularity in the middle has been smoothed in such a way that the neighborhood of the singularity looks like a hyperboloid of one sheet. Adding `-0.1*z^2` leads to a neighborhood which looks like a hyperboloid of two sheets. □

It is a little more subtle to keep singularities of type A_j^- or $A_j^+, j > 1$, while deforming others. Forgetting about the sign for a moment, these singularities have the equation $x^{j+1} + y^2 + z^2$ in a suitable coordinate system. $A_j, j > 1$, singularities are characterized by the property that their tangent cone is of degree two and consists of the union of two different planes.[5]

[4] This is also the reason why the geometers of the 19$^{\text{th}}$ century called the A_1 singularities conical singularities or singularities of type C_2. Other names are *proper node*, *ordinary double point*.

[5] This is the reason why the classical geometers called a singularity of type A_j a biplanar node B_{j+1}. A singularity whose tangent cone consists of a single multiple plane was called a uniplanar node.

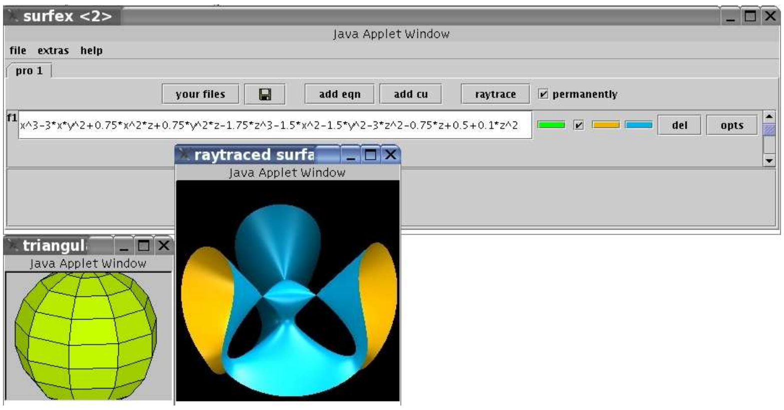


Fig. 7. Smoothing one of the four singularities of the cubic surface KM_{27}.

Let f be a polynomial in three variables x, y, z defining a singularity of type $A_j, j \geq 2$, at the origin. By the finite determinacy theorem (see, e.g., [5]), we can add an element of the ideal $I := \mathfrak{m}^2 \cdot J_f$ to f without changing the type of the singularity. Here, $\mathfrak{m}$ denotes the maximal ideal (x, y, z) of the origin and thus $\mathfrak{m}^2 = (x^2, xy, xz, y^2, yz, z^2)$. $J_f := (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$ is the so-called jacobian ideal generated by the partial derivatives of f.

Example 6. We take the singularity of type A_3^- at the origin, defined by $f := x^4 + y^2 - z^2 = 0$. Its jacobian ideal is $J_f = (x^3, y, z)$. If we choose $g_1 := xy \in \mathfrak{m}^2$ and $g_2 := y \in J_f$ we get $g := g_1 g_2 = xy^2$. Then $f + g$ still defines a singularity of type A_3 at the origin. Furthermore, $f + \varepsilon g$ is an A_3^- singularity for ε small enough.

We now come to the global situation of a cubic surface f with only isolated singularities of type $A_j, j \geq 1$. The following example describes how to use the techniques above to deform some of its singularities while keeping others:

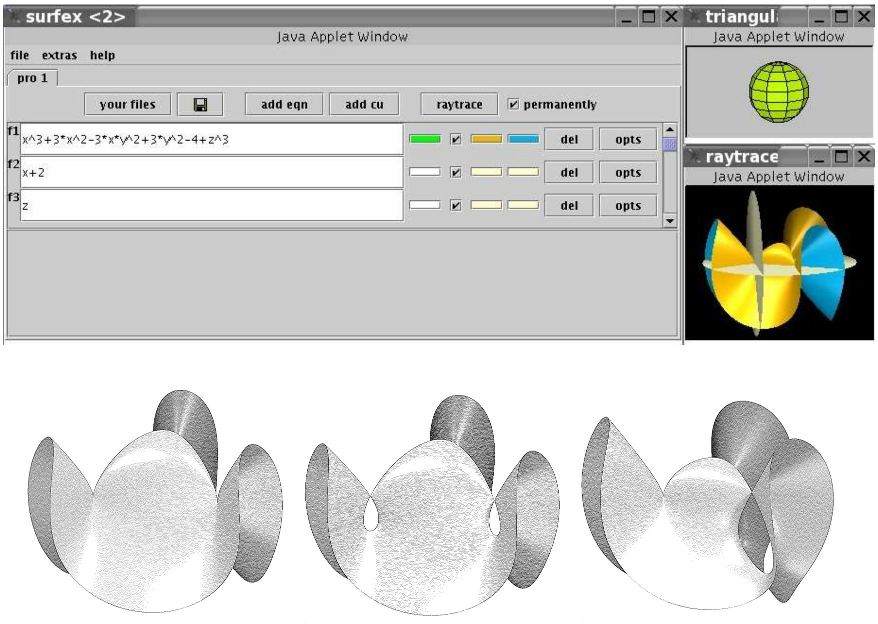
Example 7 (Deforming two of three A_2^- Singularities to A_1^- Singularities). We start with the surface KM_{43} which has exactly three singularities of type A_2^- (fig. 8(a)). The surface $tl + z^3 + z^2$ (fig. 8(b)) has three singularities of type A_1^- at the same coordinates, because the tangent cone is a cone of the form $x^2 - y^2 + z^2$ locally at each of these points. One of these singularities has the coordinates $Q := (-2, 0, 0)$. To get a surface with a singularity of type A_2^- at Q and two singularities of type A_1^-, we need to adjust the construction slightly.

Our general remarks from the beginning of this subsection tell us that we have to look at the jacobian ideal $J_{\mathrm{KM}_{43}}$ at Q. Over the rational numbers, Singular gives the following primary decomposition: $J_{\mathrm{KM}_{43}} = (x, y, z^2) \cap (x - 1, y^2 - 3, z^2) \cap (x + 2, y, z^2)$. Locally at Q, the relevant primary component is

$(x+2, y, z^2)$. We choose $E := x+2 \in (x+2, y, z^2)$. As $z^2 \in \mathfrak{m}^2$, we then know that $\mathrm{KM}_{43} + z^2 \cdot E$ has a singularity of type A_2 at Q.

Locally at the other two singularities (which both have x-coordinate 1), E takes the value $1+2 = 3$. Thus, at these singularities, $\mathrm{KM}_{43} + z^2 \cdot E$ behaves like $\mathrm{KM}_{43} + z^2 \cdot 3$, which has A_1^- singularities at these points as already seen above.

To check that our choices of planes and constants were reasonable and to understand the construction a little better, we can again use SURFEX. We type the equation of KM_{43} into SURFEX as `f1`. Then we add another two equations using the `add eqn` button and choose `f2` to be $x+2$ and `f3` to be z. If the



(a) (b) (c)

Fig. 8. Deforming the surface KM_{43} (image (a)) with three singularities of type A_2^- into KM_{28} (image (c)) with one such singularity and two A_1^- singularities.

`permanently` checkbox is activated we already see the three surfaces in one picture. When adjusting the colors by clicking at the right of the equations, we get a result similar to fig. 8. We can hide some of the surfaces by deselecting the checkbox at the right of the equations. When typing into `f1` the changes described above, we obtain successively the three lower images shown in the figure. We can produce the black/white images used for the present publication in the following way: We press the button showing the small disk, select the `dithered` checkbox, choose an appropriate resolution, and then click on save. A small dialog shows up, where we can give some filename. The high-resolution image is then computed on the webserver. From there, it can then be downloaded using the your files button in the SURFEX window. □

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