World record surfaces

Algebraic surfaces with many singularities

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2014

Abstract

Algebraic surfaces can be smooth, or they can have several sharp peaks, known as isolated singularities. It was proved already in the 19th century that there are only finitely many isolated singularities on every surface of a specific so-called degree. So the question naturally arises: how many?

World record surfaces of degrees 2 through 6, with respectively 1, 4, 16, 31, 65 singularities.

To answer the question, one must find for every degree (a natural number d) a number $\mu(d)$ such that 1) there exists a surface of degree d with $\mu(d)$ singularities, and 2) one can show that a surface of that degree could not have more singularities.

Every surface of degree d that has more singularities than any previously known surface of that degree is a new world record. If one can additionally prove that such a surface could not have more singularities, then one has shown that the world record can never be bettered!

We will see that much interesting mathematics, especially geometry, comes into play in the search for world record surfaces; both Platonic solids and the golden ratio appear multiple times, as well as finite number systems.

Introduction

We will describe some of the current world record surfaces with many singularities, as well as their often fascinating history and geometry. In general an ***algebraic surface*** of degree d is the set of all zeros of a polynomial of degree d in three variables. We begin with the simplest cases of degree 1 and 2, even though here the geometry of surfaces related to their singularities is not as interesting as for the surfaces of higher degree. From degree 3 and 4 onwards we will see connections to other parts of geometry, for example the Platonic solids and the golden ratio.

As mentioned in the abstract, one can sometimes even show that the current world records will never be bettered. This is currently the case for $d = 1, 2, 3, 4, 5, 6$. Only from degree $d = 7$ onwards is it still unclear whether the current world record (99 for $d = 7$) is also the best possible number. The following table gives an overview for low d; as described in the abstract, $\mu(d)$ means the maximum possible number of singularities on a surface of degree d:

d	1	2	3	4	5	6	7	8	d
$\mu(d) \geq$	0	1	4	16	31	65	99	168	$\approx \frac{5}{12}d^3$
$\mu(d) \leq$	0	1	4	16	31	65	104	174	$\approx \frac{4}{9}d^3$

We must note that $\mu(d)$ means more precisely the maximum number of ***complex singularities*** of an algebraic surface, the defining polynomial of which having possibly complex coefficients. However, as to date no better upper or lower bounds for a real variant $\mu_{\mathbb{R}}(d)$ of $\mu(d)$ is known, we will not go into more detail about the difference between $\mu_{\mathbb{R}}(d)$ and $\mu(d)$ here. Remarkably, all surfaces that realize the lower limits given in the table have exclusively real singularities, which makes it very easy to visualize them!

A substantially more detailed and mathematically rigorous exposition of the topic *Hypersurfaces with many singularities* can be found in the author's PhD thesis [Labs 2005]. Here we will try to explain some features of the relevant mathematics and the geometric constructions in a more visual and detailed way than in that work.

1 Planes

The geometry of a plane (a surface of degree 1) is not particularly interesting. Nevertheless we will examine these simplest surfaces in detail, as one can show in this instance with very elementary calculations that these surfaces cannot have any singularities.

We begin with the equation of a plane:

$$E: \quad ax + by + cz + d = 0,$$

where a, b, c, d have a specific constant value for every plane. A ***singularity*** of a surface is a point for which both the polynomial defining the surface and its partial derivatives are zero. If we calculate the latter, we find:

$$\frac{\partial E}{\partial x} = a, \quad \frac{\partial E}{\partial y} = b, \quad \frac{\partial E}{\partial z} = c.$$

In order for all three to vanish simultaneously, we must have $a = 0$, $b = 0$ and $c = 0$. So if the plane E does have a singularity, then the plane is of the form: $d = 0$. But such a plane only contains points when the constant d is exactly 0. The resulting equation of the plane is however $0 = 0$, which imposes no constraints at all on the variables x, y, z and therefore describes the whole of space and not a plane at all. Thus we have proved that a plane can have no singularities:

Theorem 1. *A plane has no singularities, so that in particular:*

$$\mu(1) = 0.$$

2 Quadrics

Surfaces of degree 2 are called quadrics. It is already not so easy to calculate how many singularities they can have in an elementary way, though the Greeks understood most of their properties over 2000 years ago. For a very visual presentation of many more properties than we can discuss here, see [Hilbert and Cohn-Vossen 1952].

Today as a mathematics student one usually learns about quadrics in a course on linear algebra: the nearly complete understanding of quadrics forms one of the highpoints of this course. We present here the theorem proven there:

Theorem 2 (Classification of quadrics)**.** *Every quadric in $\mathbb{R}^3$ can be moved into one of the following three forms through a rotation and/or translation:*

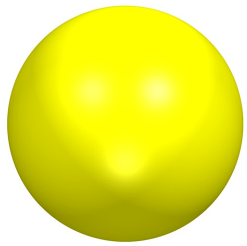
Case 1: $ax^2 + by^2 + cz^2 = 0$,

Case 2: $ax^2 + by^2 + cz^2 - 1 = 0$,

Case 3: $ax^2 + by^2 - z = 0$,

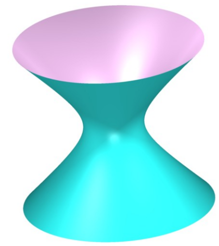
where $a, b, c \in \mathbb{R}$ are some constants. If one also allows shrinking and stretching, one can manage that $a, b, c \in \{-1, 0, 1\}$.

The most important cases in this classification can be seen on the next page. In most cases, one can understand the geometry of a quadric rather well from its equation. For example, one can set one of the three variables to a constant value; this then gives a



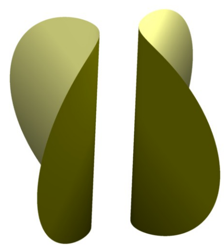
Sphere (surface of a ball)
(case 2: $a, b, c = 1$)

Ellipsoid
(case 2: $a, b, c > 0$)

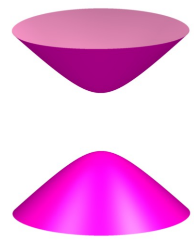


Hyperboloid of one sheet
(case 2: $a, c > 0,\ b < 0$)

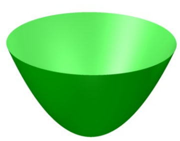
(Elliptic) cylinder
(case 2: $a, b > 0,\ c = 0$)



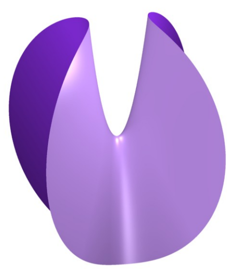
Hyperbolic cylinder
(case 2: $a > 0,\ b < 0,\ c = 0$)



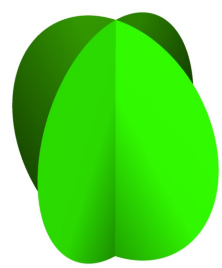
Hyperboloid of two sheets
(case 2: $a > 0,\ b, c < 0$)



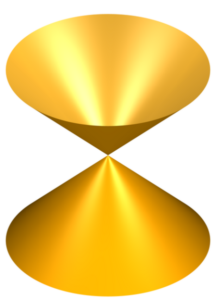
Elliptic paraboloid
(case 3: $a, b > 0$)



Hyperbolic paraboloid
(case 3: $a > 0, b < 0$)



Two planes
(case 1: $a > 0,\ b < 0,\ c = 0$)



Double cone
(case 1: $a, b > 0, c < 0$)

A point
(case 1: $a, b, c > 0$)

A line
(case 1: $a, b > 0,\ c = 0$)

The classification of quadrics.

plane curve of degree two, so either an ellipse, a hyperbola, a parabola, two lines, etc. — try it!

From the pictures of the classification one can already divine that a quadric can have at most one isolated singularity, but we can also very easily prove this with the help of the above classification theorem:

Theorem 3. *A quadric has at most one isolated singularity*[1]*. This is an ordinary double point*[2]*. In particular:*

$$\mu(2) = 1.$$

Proof: Because of Theorem 2 on the classification of quadrics we only have to look at the cases given there, calculate the partial derivatives for each, and determine the possible singularities. We consider here only the cases of Theorem 2 with $a, b, c \neq 0$:

Case 1: In the first case the partial derivatives are $2ax, 2by, 2cz$. For $a, b, c \neq 0$ these can only be simultaneously zero when $x = y = z = 0$. The point $(0, 0, 0)$ is indeed a point of the quadric, since $a \cdot 0^2 + b \cdot 0^2 + c \cdot 0^2 = 0$, and therefore the only singularity of the quadric. As we saw above this is an ordinary double point.

Case 2: In the second case the partial derivatives are also $2ax, 2by, 2cz$. For $a, b, c \neq 0$ these can again only be simultaneously zero when $x = y = z = 0$. The point $(0, 0, 0)$ is however not a point of the quadric, since $a \cdot 0^2 + b \cdot 0^2 + c \cdot 0^2 = 0 \neq 1$. Thus in this case the quadric has no singularities.

Case 3: In the last case the partial derivatives are $2ax, 2by, -1$. The partial derivative with respect to z is constantly -1 and can thus never be 0, so these quadrics also can have no singularities.

Thus altogether singularities can only arise in the first case. □

In fact, quadrics may also have infinitely many singularities, but then the singularities are not isolated, i.e. there is no space between them. Consider the example $f : x^2 - y^2 = 0$ (i.e. case 1 where $a = 1$, $b = -1$, $c = 0$). The zero-set of f consists of two planes because $x^2 - y^2 = 0 \Leftrightarrow (x - y)(x + y) = 0$ which is equivalent to $y = x$ or $y = -x$. The singularities of f can be computed easily: $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2y$, $\frac{\partial f}{\partial z} = 0$. Thus the singularities are exactly all the points $(0, 0, t)$ for $t \in \mathbb{R}$. This is the line of intersection of the two planes $y = x$ and $y = -x$, as can be seen in the panel on the previous page.

[1] An ***isolated singularity*** is a singularity for which a neighborhood exists that contains no other singularities. In this article, the word 'singularity' is used a little imprecisely, generally meaning 'isolated singularity' but occasionally meaning exactly 'singularity' depending on the context.

[2] An ***ordinary double point*** is a singularity for which, in the neighborhood of such a point, the surface looks similar to one of the two left images in the bottom row on the previous page.

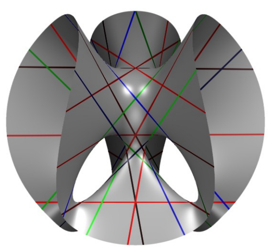
3 Cubic surfaces and Platonic solids

We could easily calculate by hand that planes can have no singularities. Already for quadrics we had to use a result from a standard course in order to prove that these surfaces can have at most one singularity.

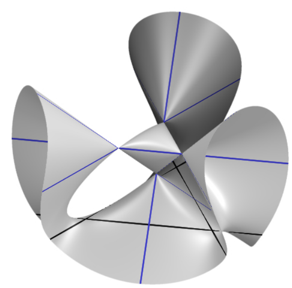
For cubics (that is, surfaces defined by a polynomial of degree 3) the situation is more complicated. In order to answer the question of how many singularities are possible here, we must reach for a result that one encounters first in special courses on algebraic geometry — which however the mathematicians of the 19th century already knew.

3.1 The history of cubic surfaces

The first interesting results about cubic surfaces were found in 1849 by the two British mathematicians George Salmon and Arthur Cayley in a letter exchange: exactly 27 lines lie on every cubic surface with no singularities (left in the figure below) — and if one transforms the surface so that singularities appear, then some of the lines merge into one another (right in the figure below).



Clebsch's cubic has no singularities and 27 lines.



Cayley's cubic has 4 singularities and only 9 different lines.

This may seem rather amazing at first sight, since cubic surfaces look very curved and not at all straight, but one can really prove it; under the assumption that at least one line lies on such a surface, it is even not so hard to determine the true number of lines.

It is impressive that cubic surfaces are still the subject of research, even though so much is already known about them; this is mainly because small parts of cubic surfaces are used today to approximate more complicated objects. This is applied for example in the field of Computer Aided Design, as cubic surfaces have many useful properties, such as being parameterisable. Unfortunately we cannot expand on this here; we only want to illuminate a small detail of the very extensive theory of cubic surfaces, namely the question of their maximum number of singularities. The website [Labs and van Straten 2000] gives a broad overview of all kinds of cubic surfaces; see also [Holzer and Labs 2006].

3.2 An upper bound for $\mu(d)$

Already in the 19th century mathematicians associated to given algebraic surfaces other algebraic surfaces which had a special relationship to them. One such associated surface is the so-called ***dual surface***. It has a great deal to do with the original surface; in particular, the dual surface of the dual surface is the original surface again! Though we cannot explain the concept of dual surface here, we can at least learn to use a formula for its degree: if the degree d of a given algebraic surface f is at least $d \geq 3$ and if we denote the degree of the dual surface of f with d^*, then:

$$d^*(f) \leq d(d-1)^2 - 2\mu(f),$$

where $\mu(f)$ is the number of isolated singularities of f. Since it is known that the dual surfaces of quadrics are again quadrics, we get that $d^*(f) \geq 3$ if $d \geq 3$. If we insert this into the above inequality and rearrange, we get the following upper bound for the number $\mu(f)$ of singularities on f, and since f was of arbitrary degree d, also an upper bound for $\mu(d)$:

Theorem 4 (19th century, possibly by G. Salmon)**.** *The following inequality holds for the maximum possible number $\mu(d)$ of isolated singularities on a surface of degree $d \geq 3$:*

$$\mu(d) \leq \frac{1}{2}\big(d(d-1)^2 - 3\big).$$

If we put in this formula certain values for $d \geq 3$, then we find for example: $\mu(3) \leq 4$, $\mu(4) \leq 16$, $\mu(5) \leq 34$. We will see later that in fact $\mu(4) = 16$ holds, but that the true bound for degree 5 is lower than 34. But first back to degree 3.

3.3 The Cayley cubic with four singularities

We have just seen that a cubic surface can have at most four singularities, but does a surface that attains this bound really exist? Yes, it does; to honor a mathematician who worked a good deal on cubic surfaces, it is often called the ***Cayley cubic*** today, though the Swiss mathematician Ludwig Schläfli was the first to study the singularities of cubic surfaces in detail in 1863.

One gets a particularly symmetric equation for a cubic surface with the maximal possible number of $\mu(3) = 4$ singularities if one takes for example the equation $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+\frac{1}{w} = 0$, multiplies through by the common denominator, and sets $w = 1-x-y-z$:

$$\text{Cay}\colon\ yzw + xzw + xyw + xyz = 0, \quad w = 1-x-y-z.$$

A good exercise for the reader is to verify that this surface does indeed have exactly four singularities as claimed. These are the four points $(0,0,0)$, $(0,0,1)$, $(0,1,0)$ and $(1,0,0)$. This brings us the:

Theorem 5 (middle of the 19th century)**.** *A cubic surface can have at most four isolated singularities:*

$$\mu(3) = 4.$$

3.4 The Cayley cubic and tetrahedral symmetry

Here it should not surprise us that after permuting the coordinates of one of these points one again gets one of the singularities — after all, the equation of the surface Cay remains completely preserved when x, y, z are permuted. In such a case one says that the equation is ***invariant*** under the permutations. Let us examine these permutations somewhat more closely; they are the following five:

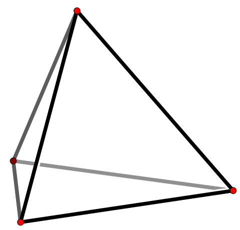
$$\begin{array}{ccc@{\quad}ccc@{\quad}ccc@{\quad}ccc@{\quad}ccc}
 & \alpha: & & & \beta: & & & \gamma: & & & \delta: & & & \eta: & \\
x & \mapsto & y & x & \mapsto & y & x & \mapsto & z & x & \mapsto & z & x & \mapsto & x \\
y & \mapsto & x & y & \mapsto & z & y & \mapsto & y & y & \mapsto & x & y & \mapsto & z \\
z & \mapsto & z & z & \mapsto & x & z & \mapsto & x & z & \mapsto & y & z & \mapsto & y.
\end{array}$$

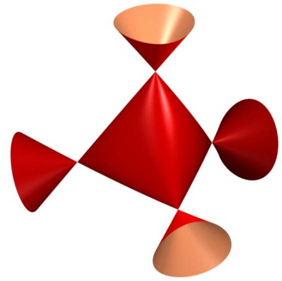
If we also count the "permutation" ι that does not perform any permutation (i.e., that takes x to x, y to y, and z to z) to the total, then we have 6 permutations altogether. This is no coincidence, as one can easily prove that there in fact exist exactly $n! = n(n-1)(n-2)\cdots 1$ permutations of n different letters: for the first letter one namely has n image letters to choose from, for the next only $n-1$, then only $n-2$ etc.

Permutations have the feature that one can carry out two arbitrary permutations α and β of a given number of letters in succession (written $\beta \circ \alpha$) and one will again get a permutation. In addition one can obviously undo a permutation α using its so-called ***inverse permutation*** α^{-1}. Lastly it holds that $(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)$ for three arbitrary permutations. A set of permutations with all these properties is called a ***group***; the permutations of a given set of n different letters thus form a group, the so-called ***symmetric group on*** n ***letters***, denoted by S_n.

In the above example of the group of permutations under which the equation of the surface Cay remained invariant, we can understand all this very concretely: if we first apply α to the three variables (x, y, z), then x and y exchange places, (y, x, z); if we apply α again, then we come back to the original order: (x, y, z), since in this case $\alpha = \alpha^{-1}$. This does not hold for β, as $\beta(x, y, z) = (y, z, x)$ and $\beta(y, z, x) = (z, x, y)$; in the end one does have $\beta(z, x, y) = (x, y, z)$, as the threefold application of β is the same as doing nothing, i.e. $\beta \circ \beta \circ \beta = \iota$. One can easily check that all six permutations $\alpha, \beta, \gamma, \delta, \eta, \iota$ can be created from the repeated application of α and β. One thus says that α and β ***generate*** the group.

If now in the above equation of Cay we replace the x, y, z, w by certain planes $\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w}$, one can even achieve that the equation Cay remains invariant under arbitrary permutations of the four "letters" $\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w}$, namely, when $\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w}$ describe the four faces of a regular tetrahedron.

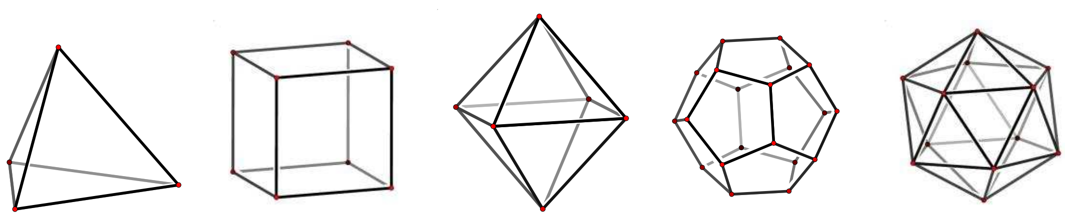




A regular tetrahedron and the tetrahedrally symmetric version of the Cayley cubic.

Altogether these form a group of $4! = 24$ permutations under which Cay is invariant. Have a look at the resulting tetrahedrally symmetric surface in the figure above. Naturally you can also easily compute the coordinates of the singularities here, though perhaps the calculations become slightly boring. Because of the chosen tetrahedral symmetry, exactly the 4 vertices of the tetrahedron with faces $\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w}$ should result.

The other Platonic solids cannot be used in a similar way for the construction of cubic surfaces. However, since some of these solids, which have fascinated mathematicians for over 2000 years, appear later, we briefly give an overview in the figure below. The Platonic solids also have much deeper relationships with singularities which we unfortunately cannot go into here; see for example [Greuel 1992].



Tetrahedron Cube Octahedron Dodecahedron Icosahedron

The Platonic solids.

4 The Kummer quartic and tetrahedra again

Just one year after Ludwig Schläfi classified cubic surfaces with respect to their singularities in 1863, the German mathematician Eduard Kummer established the maximum possible number $\mu(4)$ of singularities on a surface of degree 4 (so-called quartics). As we saw in Theorem 4, it holds that $\mu(4) \leq 16$. Kummer first noted that the so-called Fresnel wave surface actually has 16 singularities, thus giving:

Theorem 6 (E. Kummer, 1864)**.** *A quartic can have maximally* 16 *isolated singularities:*

$$\mu(4) = 16.$$

But he did not stop with this; he made detailed studies of quartics that have the maximal number of 16 singularities. He also gave a very beautiful tetrahedrally–symmetric family of equations for such surfaces:

$$\mathrm{Ku}_\mu := \left(x^2+y^2+z^2-\mu^2\right)^2 - \lambda\, \tilde{x}\tilde{y}\tilde{z}\tilde{w}, \quad \lambda = \frac{3\mu^2-1}{3-\mu^2}, \quad \mu \in \mathbb{R},$$

where $\tilde{x}, \tilde{y}, \tilde{z}, \tilde{w}$ are the faces of a regular tetrahedron as used in the section on tetrahedrally–symmetric cubics and μ is a real number (in the figure below $\mu = 1.3$).

A Kummer quartic with 16 ordinary double points.

A wonderful book about these quartics with the maximal number of singularities, which today are named after Kummer, is [Hudson 1905]. From the fact that whole books have been written about these surfaces, one sees that we could say much more about them here; since we do not have enough space we must however unfortunately refer to the indicated literature.

5 Togliatti quintics, pentagons, and the golden ratio

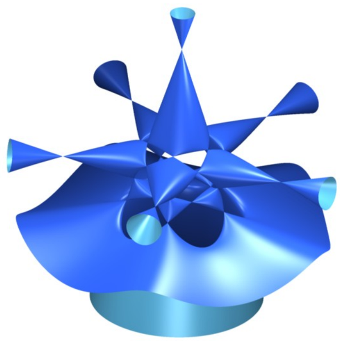
The Italian mathematician Eugenio Giuseppe Togliatti proved already in 1937 that a surface of degree 5 (hence the name quintic) exists with 31 singularities — at that time a world record!

Since as already mentioned the upper bound in Theorem 4 only proves that there can be no more than 34 singularities on a quintic, and since in the meantime nobody could find a better upper bound, geometers searched for decades for a surface of degree 5 with at least 32 singularities, until finally in 1980 the French mathematician Arnaud Beauville was able to show that a quintic cannot have more than 31 singularities through an interesting relationship with the theory of codes. This means that Togliatti's world record can never be bettered! Thus it holds that:

Theorem 7 (E.G. Togliatti, 1937; A. Beauville, 1980)**.** *A quintic can have maximally* 31 *isolated singularities:*

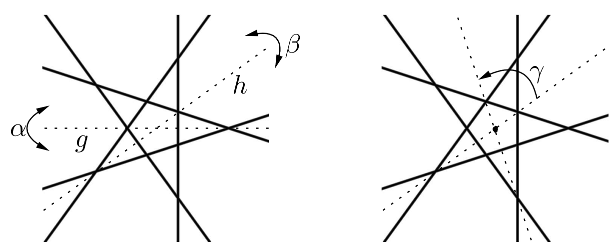
$$\mu(5) = 31.$$

Unluckily, Togliatti's construction is not so easy to visualise, so for the figure below we use a construction of a surface by the German mathematician Wolf Barth from the 1990s which also gives 31 ordinary double points. Like in the construction of the tetrahedrally symmetric Cayley cubic, it may seem natural to use a Platonic solid here. However, since there is no Platonic solid that has either exactly five faces or exactly five planes of symmetry, it is not so clear how one could construct this surface of degree 5 in this way.



Barth's Togliatti quintic with 31 ordinary double points.

Therefore the pictured quintic with 31 singularities has less symmetry, namely the symmetry of a plane pentagon, i.e. under all reflections in the x, y–plane that hold fixed a regular plane pentagon, Barth's Togliatti quintic is also fixed. On the basis of the figure below one can easily convince oneself that the group of all permutations of points of the plane that hold the pentagon fixed consists exactly of 10 permutations, namely five rotations (where the rotation by $0°$ is included) and five reflections.



Reflections and and rotations that hold a regular pentagon fixed: α is the reflection in the line g, β that in h, and γ is the rotation around the origin by the angle $\frac{1}{5} \cdot 360°$.

The equation of the quintic pictured above is not easy to find; Barth started with a family of equations that depends on three parameters a, b, d, namely

$$\mathrm{Bar}_{a,b,d} := P - az \cdot Q^2,$$

where P is a polynomial of degree 5 and Q is a certain quadric, specifically:

$$\begin{aligned}
P &:= \textstyle\prod_{j=0}^{4}\left(\cos\left(\frac{2\pi j}{5}\right)x+\sin\left(\frac{2\pi j}{5}\right)y-1\right)\\
&= \tfrac{1}{16}\Big(x^5-5x^4-10x^3y^2-10x^2y^2+20x^2\\
&\qquad\qquad +5xy^4-5y^4+20y^2-16\Big),\\
Q &:= x^2+y^2+bz^2+z+d.
\end{aligned}$$

Through geometric and algebraic arguments, Barth eventually found values for a, b, d that indeed give a surface with 31 ordinary double points:

$$a=-\frac{5}{32},\quad b=-\frac{5-\sqrt{5}}{20},\quad d=-(1+\sqrt{5}).$$

Nr. 23 on the website [Bothmer and Labs 2006] (Calendar.AlgebraicSurface.net) shows a film which runs through various values for a and b, eventually reaching the values at which 31 singularities arise.

The number d does not appear here coincidentally; rather it is closely related to the so–called ***golden ratio***. A division of a length is in the golden ratio, when the greater part is in the same ratio to the whole as the smaller part to the greater.

$$\underbrace{\qquad\qquad\qquad\qquad}_{1-x}\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{x}$$

The golden ratio x satisfies $\frac{x}{1}=\frac{1-x}{x}$.

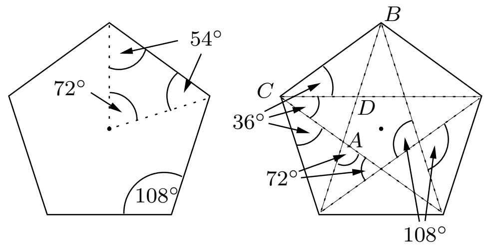
If the whole length is 1, then we have:

$$\begin{aligned}
\frac{x}{1} &= \frac{1-x}{x}\\
\iff 0 &= x^2+x-1=\left(x+\frac{1}{2}(1+\sqrt{5})\right)\left(x+\frac{1}{2}(1-\sqrt{5})\right)\\
\iff x &\in \left\{-\frac{1}{2}\pm\frac{1}{2}\sqrt{5}\right\}.
\end{aligned}$$

Since a length x cannot be negative, we find the single solution $x=-\frac{1}{2}+\frac{1}{2}\sqrt{5}=0.618\ldots$. Another remarkable fact about this number is that

$$\frac{1}{x}=x+1=1.618\ldots,$$

which comes directly from the defining equation. Geometrically the golden ratio appears in pentagons, so it is not surprising that the parameter d that Barth found is also related to it. In fact, one can rather easily prove that the diagonals of a regular pentagon divide each other in the golden ratio.



The diagonals of a regular pentagon divide each other in the golden ratio.

Since in this figure the triangles ABC and ADC are similar, the following relationship holds for the line segments between these points:

$$\frac{|AC|}{|AB|} = \frac{|AD|}{|AC|}.$$

In addition $|AD| = |AB| - |AC|$ and $|AC| = |BD|$; from this we get:

$$\frac{|BD|}{|AB|} = \frac{|AB| - |BD|}{|BD|}.$$

Thus we have proved that the point D divides the line AB in the golden ratio. It is not surprising that, since Barth's construction of a quintic with 31 singularities uses pentagons and pentagonal symmetry, the golden ratio also appears in one of the three parameters, namely d:

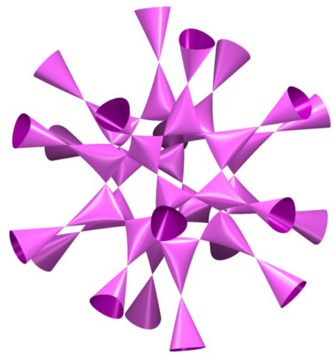
$$d = -2 \cdot \frac{1}{2}(1 + \sqrt{5}) = -2 \cdot \tau,$$

where $\tau := x + 1 = \frac{1}{x} = \frac{1}{2} + \frac{1}{2}\sqrt{5}$ is, as above, the reciprocal of the golden ratio.

6 The icosahedrally symmetric Barth sextic

The Barth sextic has a very remarkable history. Already since the early 1980s it was known that surfaces of degree 6 can not have more than 66 singularities. In 1982 there appeared an article in which the authors claimed to prove that sextics which are essentially of the form $P - az{\cdot}Q^2 = 0$ can have at most 64. Mathematical articles that appear in respected journals (like this one) can however sometimes have errors — and that was indeed the case here. For in 1996 Wolf Barth constructed a surface of degree six that is exactly of the form given above and has 65 singularities (see the figure and also Nr. 6 on the website [Bothmer and Labs 2006]; though in the figure one can only see 50 of the 65 total singularities)!

Almost at the same time the two mathematicians Jaffe and Ruberman additionally managed to prove that 66 singularities are not possible, and thus also that Barth's world record is unbeatable:



The Barth sextic with 65 ordinary double points.

Theorem 8 (W. Barth 1996, D.B. Jaffe / D. Ruberman 1997)**.** *A sextic can have maximally* 65 *isolated singularities:*

$$\mu(6) = 65.$$

One can already divine the icosahedral symmetry of Barth's construction from the figure. The exact equation of the surface is:

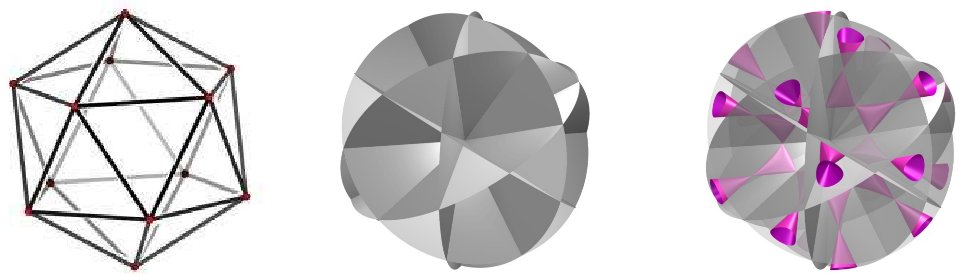
$$\mathrm{Bar}_{65}\colon\ P_6 - \alpha K^2 = 0,$$

where P_6 represents the six planes through the origin which lie orthogonal to the six diagonals of the regular icosahedron and fulfill the equation $P_6 := (\tau^2 x^2 - y^2)(\tau^2 y^2 - z^2)(\tau^2 z^2 - x^2)$ with $\tau := \frac{1}{2}(1 + \sqrt{5})$. In addition $K := x^2 + y^2 + z^2 - 1$ describes the sphere with radius 1 and the parameter α is $\alpha := \frac{1}{4}(2\tau + 1) = \frac{1}{4}(2 + \sqrt{5})$. Interestingly the reciprocal τ of the so–called golden ratio, which was already relevant to the quintic with 31 singularities, appears here in multiple places as well.

Using the equation one can see that the entire surface is invariant under the whole symmetry group of the icosahedron, since this naturally holds for the six planes P_6 and also the sphere K. For other values of α the surface is still icosahedrally symmetric but has fewer singularities: try it in SURFER [Greuel et al. 2012]!

7 A septic with 99 singularities and finite number systems

As with the quintic, in the case of surfaces with degree 7 one cannot use Platonic solids in an obvious way to construct a surface with many singularities. Therefore one can first try to use a regular n-agon again (in this case of course a heptagon), but to start with this gives too large a search space.



A regular icosahedron. Its 6 planes of symmetry. The planes & Barth's sextic.

The construction of Barth's sextic.

In order to find in this huge search space the small point (or the few small points) which will give a septic with very many singularities, the author of this article used in his dissertation [Labs 2005] the fact that one can have algebraic surfaces over other number systems than the real numbers. We already mentioned in the introduction that one can study algebraic surfaces over the complex numbers. Now we will go to another kind of number system, namely the finite number systems. We all know such finite number systems from clocks:

$$23{:}00 + 3 \text{ hours } (= 26{:}00) = 2{:}00.$$

These calculations in such finite number systems function particularly well when we use exactly p numbers, where p is a prime number. The numbers used are thus:

$$0, 1, 2, \ldots, p-1$$

and it holds that

$$p = 0,$$

exactly like on clocks 24:00 = 0:00. For a prime number p, such a number system is usually designed as $\mathbb{F}_p$ and is called the ***finite field with*** p ***elements***. In $\mathbb{F}_p$ one can not only add but also multiply and divide wonderfully. With a clock ($24 = 3 \cdot 2 \cdot 2 \cdot 2$ is not a prime number!) on the contrary one has the problem that

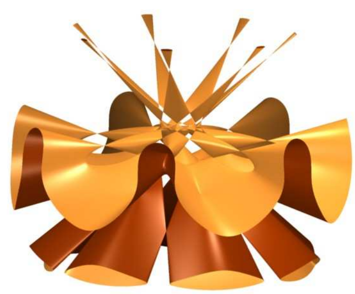
$$3{:}00 \cdot 8 = 24{:}00 = 0{:}00;$$

in $\mathbb{F}_p$ it can never happen that the product of two numbers which are not zero gives zero, which substantially simplifies calculating; also one can therefore divide by each of the finitely many numbers (apart from 0). Calculating in finite fields has countless applications; in particular coding theory and cryptography have become important recently — no cell phone would be possible without finite fields.

We cannot go more deeply into this here, but we at least want to see an example of a curve over a finite field, specifically the circle $c\colon\ x^2 + y^2 = 1$ over the field $\mathbb{F}_3$, in which there exist only the three numbers $0, 1, 2$. In $\mathbb{F}_3$ we have $1 + 1 + 1 = 0$, so $1 + 2 = 0$, and thus that we can also write -1 for 2. We now search for all points (a, b) with coordinates in the finite field $\mathbb{F}_3$ which satisfy the equation $a^2 + b^2 = 1$. Luckily there are only $3 \cdot 3 = 9$ different points (a, b) with coordinates in $\mathbb{F}_3$.

- Let's begin with the point $(0, 0)$; we have $0^2 + 0^2 = 0 \neq 1$, thus $(0, 0)$ is not a point on the circle c.
- Now the point $(1, 0)$: we have $1^2 + 0^2 = 1$, thus $(1, 0)$ and naturally also $(0, 1)$ are points on c.
- Now for $(2, 0)$: $2^2 + 0^2 = 4 = 1$, thus $(2, 0)$ and also by symmetry $(0, 2)$ are points on c.
- For $(1, 1)$ we get $1^2 + 1^2 = 2 \neq 1$, thus $(1, 1)$ does not lie on the circle.
- For $(2, 1)$ and $(1, 2)$ we find $2^2 + 1^2 = 5 = 2 \neq 1$; thus both of these points also do not lie on c.
- Finally $(2, 2)$: $2^2 + 2^2 = 8 = 2 \neq 1$; this point also does not lie on c.

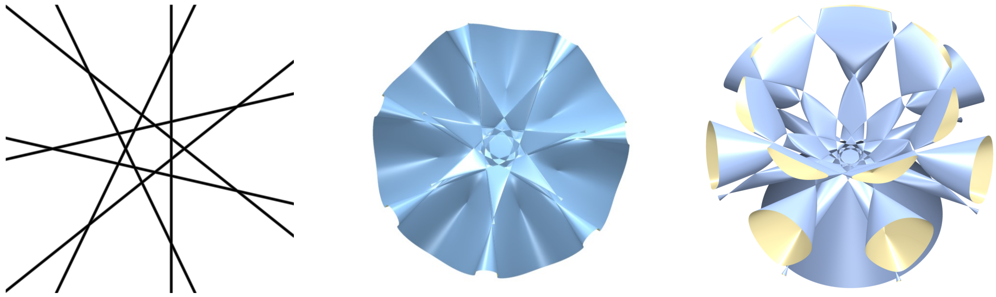
So out of the 9 points (a, b) with coordinates in $\mathbb{F}_3$, the points $(1, 0)$, $(0, 1)$, $(2, 0)$ and $(0, 2)$ lie on c, but not the others. As one sees finding solutions over finite fields is very easy, because one can try all the possibilities!



A septic with 99 ordinary double points.

In a similar way and using several more algebraic and geometric arguments, in 2004 the author of this article constructed a surface of degree 7 (a septic) with 99 singularities using the computer algebra program SINGULAR [DGPS 2012], which is particularly good at applications to algebraic geometry and singularities.

The equation of this septic is quite similar to the quintic with 31 singularities that Barth constructed: the surface has the symmetry of a regular heptagon; one can see this particularly well from 'above':



A regular heptagon, and the septic with 99 singularities, seen once from 'above' and once somewhat slanted and zoomed in.

The equation of the septic with 99 singularities takes a little while to describe. First let α be a solution of:

$$7\alpha^3 + 7\alpha + 1 = 0. \tag{1}$$

Then the surface S_α of degree 7 with equation $S_{99} := S_\alpha := P - U_\alpha$ has exactly 99 ordinary double points and no further singularities, where P and U_α are defined in the following way ($w = 1$):

$$\begin{aligned} P &:= x\cdot\left[x^6 - 3\cdot 7\cdot x^4 y^2 + 5\cdot 7\cdot x^2 y^4 - 7\cdot y^6\right] \\ &\quad + 7\cdot z\cdot\Big[\big(x^2+y^2\big)^3 - 2^3\cdot z^2\cdot\big(x^2+y^2\big)^2 + 2^4\cdot z^4\cdot\big(x^2+y^2\big)\Big] - 2^6\cdot z^7, \\ U_\alpha &:= (z + a_5 w)\big((z+w)(x^2+y^2) + a_1 z^3 + a_2 z^2 w + a_3 z w^2 + a_4 w^3\big)^2, \end{aligned}$$

$$\begin{aligned} a_1 &:= -\tfrac{12}{7}\alpha^2 - \tfrac{384}{49}\alpha - \tfrac{8}{7}, & a_2 &:= -\tfrac{32}{7}\alpha^2 + \tfrac{24}{49}\alpha - 4, \\ a_3 &:= -4\alpha^2 + \tfrac{24}{49}\alpha - 4, & a_4 &:= -\tfrac{8}{7}\alpha^2 + \tfrac{8}{49}\alpha - \tfrac{8}{7}, \\ a_5 &:= 49\alpha^2 - 7\alpha + 50. \end{aligned}$$

The equation (1) has exactly one real solution $\alpha_\mathbb{R} \in \mathbb{R}$:

$$\alpha_\mathbb{R} \approx -0.14010685, \tag{2}$$

and all singularities of $S_{\alpha_\mathbb{R}}$ are also real. In the right panel of the figure above however some of the singularities cannot be seen in the pictured part because of the zoom.

The existence of the equation that the author found in 2004 shows half of the following result; the other half was proved at the beginning of the 1980s by A.N. Varchenko:

Theorem 9 (Labs 2004, Varchenko 1983)**.** *For the maximal number $\mu(7)$ of isolated singularities on a septic it holds that:*

$$99 \le \mu(7) \le 104.$$

99 singularities are currently the world record for surfaces of degree 7; to date however no reason is known why there could not in fact be a septic with 104 singularities. Therefore, it may still be possible to better the world record of the author of this article — so enjoy the search for a septic with more than 99 singularities!

References

[Bothmer and Labs 2006] BOTHMER, H.-C. G.V. and O. LABS: *Geometrical Animations Advent Calendar 2006*. `Calendar.AlgebraicSurface.net`, 2006.

[Greuel 1992] GREUEL, G.-M.: *Deformation und Klassifikation von Singularitäten und Moduln*. Jahresber. Deutsch. Math.-Verein. Jubiläumstagung 1990, pages 177–238, 1992.

[DGPS 2012] DECKER, W. and GREUEL, G.-M. and PFISTER, G. and SCHÖNEMANN, H.: SINGULAR *3-1-6 — A computer algebra system for polynomial computations.* `www.singular.uni-kl.de`.

[Hilbert and Cohn-Vossen 1952] HILBERT, D. and S. COHN-VOSSEN: *Geometry and the Imagination*. Chelsea, New York, 1952.

[Holzer and Labs 2006] HOLZER, S. and O. LABS: *in: M. Elkadi and B. Mourrain and R. Piene: Algebraic Geometry and Geometric Modeling*, chapter *Illustrating the Classification of Real Cubic Surfaces*. Springer, 2006.

[Hudson 1905] HUDSON, R.W.H.T.: *Kummer's Quartic Surface*. Cambridge University Press, 1905.

[Labs 2005] LABS, O.: *Hypersurfaces with Many Singularities*. PhD thesis, Universität Mainz, 2005. Downloadable as PDF from `www.OliverLabs.net`.

[Labs and van Straten 2000] LABS, O. and D. VAN STRATEN: *The Cubic Surface Homepage*. `Cubics.AlgebraicSurface.net`, 2000.

[Greuel et al. 2012] GREUEL, G.-M. et al.: SURFER – *A Software for the Visualization of Real Algebraic Surfaces*. `www.Imaginary-Exhibition.com/program/surfer`.